

Maxwell-independence: a new rank estimate for the 3D rigidity matroid

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Abstract

Maxwell's condition states that the edges of a graph are independent in its d -dimensional generic rigidity matroid only if (a) the number of edges does not exceed $d|V| - \binom{d+1}{2}$, and (b) this holds for every induced subgraph. We call such graphs *Maxwell-independent* in d dimensions. Laman's theorem shows that the converse holds in 2D. While the converse is false in 3D, we answer the following questions in the affirmative. The first question was posed by Tibor Jordán at the 2008 rigidity workshop at BIRS [1].

Question 1: Does *every* maximal, Maxwell-independent set of a graph have size at least the rank? *Question 2:* Is there a better and tractable combinatorial rank upper bound (than the number of edges) for Maxwell-independent graphs?

We give affirmative answers to both questions and construct a set of edges that meets the latter upper bound and contains a maximal (true) independent set. We extend these bounds to special classes of non-Maxwell-independent graphs. As one consequence, the answers also give simpler proofs of correctness for existing algorithms that give rank bounds.

1 Introduction

It is a long open problem to combinatorially characterize the 3D bar-joint rigidity of graphs. The problem is at the intersection of combinatorics and algebraic geometry, and crops up in practical algorithmic applications ranging from mechanical computer aided design to molecular modeling.

The problem is equivalent to combinatorially determining the generic rank of the 3D bar-joint rigidity matrix of a graph G . The d -dimensional bar-joint rigidity matrix of a graph $G = (V, E)$, denoted $R_d(G)$, is a matrix of indeterminates $p_1(v), p_2(v), \dots, p_d(v)$ that represent the coordinate position $\mathbf{p}(v) \in \mathbb{R}^d$ of the *joint* corresponding to a vertex $v \in V$. The matrix has one row for each edge $e \in E$ and d columns for each vertex $v \in V$. The row corresponding to $e = (u, v) \in E$ represents the *bar* from $p(u)$ to $p(v)$ and has d non-zero entries $\mathbf{p}(u) - \mathbf{p}(v)$ (resp. $\mathbf{p}(v) - \mathbf{p}(u)$), in the d columns corresponding to u (resp. v).

A subset of edges of a graph G is said to be *independent* (we drop “bar-joint” from now on) in d -dimensions, when the corresponding set of rows of $R_d(G)$ are generically independent, or independent for a generic instantiation of the indeterminate. This yields the 3D rigidity matroid associated with a graph G . The graph is *rigid* if the number of generically independent rows or the rank of $R_d(G)$ is maximal, i.e., $d|V| - \binom{d+1}{2}$, where $\binom{d+1}{2}$ is the number of rotational and translational degrees of freedom of a rigid body in \mathbb{R}^d [4].

Clearly, the number of edges of G is a trivial upper bound on the generic rank of $R_d(G)$, or alternatively the *rank of the d -dimensional rigidity matroid of G* . Thus, a graph is independent in d dimensions only if (a) $|E|$ does not exceed $d|V| - \binom{d+1}{2}$; and (b) this holds for every induced subgraph. This is called Maxwell's condition in d dimensions, and we call such graphs G *Maxwell-independent* in d dimensions. In other words, Maxwell's condition states that for any subset of edges of G , independence implies Maxwell-independence.

In 2D, the famous Laman's theorem states that the converse is also true. I.e., Maxwell-independence implies independence. So, in 2D, the Maxwell-independent subsets of edges define the same matroid as the rigidity matroid given by the independent subsets of rows of the rigidity matrix. Thus the rank of the 2D rigidity matroid of a graph G is exactly the size of a maximal, Maxwell-independent subset of edges (here, by *maximal* we mean that no edge can be added without violating Maxwell-independence). Thus all maximal, Maxwell-independent subsets of edges of G must have the same size.

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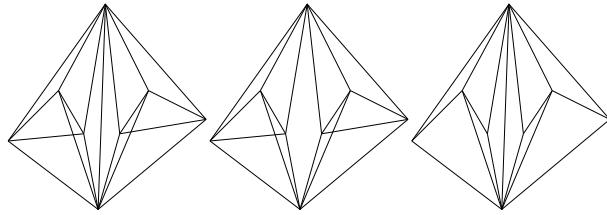


Figure 1: The graph on the left consists of two K_5 's intersecting on an edge, which is called a *double-banana*. The middle graph and the right graph are two maximal Maxwell-independent sets of the left graph and they have different sizes (the middle is of size 18 and the right is of size 17) .

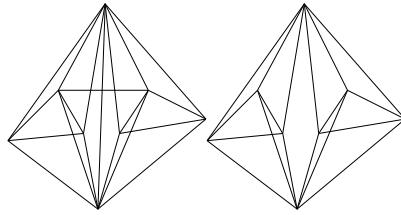


Figure 2: On the left is a double-banana-bar, which consists of a double-banana and a bar connecting two vertices from each banana. Notice that this double-banana-bar is rigid, thus every maximal independent set of it has $3|V| - 6 = 18$ edges. On the right we have a maximal Maxwell-independent set of the double-banana-bar, which has $3|V| - 6 = 18$ edges. It is well-known that the figure on the right is dependent, so every maximal independent set of it has size less than $3|V| - 6 = 18$. So the right figure cannot contain a maximal independent set of size $3|V| - 6$.

In 3D, however, different maximal, Maxwell-independent subsets of a graph may have different sizes, see Figure 1. I.e, in 3D, the set of Maxwell-independent subsets of edges does not form a matroid. Clearly, any maximal independent set of edges of G is itself Maxwell-independent, so the rank of the rigidity matroid of a graph is at most the size of *some* maximal Maxwell-independent subset of edges and this generalizes to any dimension. But this reduces to the trivial upper bound of number of edges, for Maxwell-independent graphs G . For other special classes of graphs such as graphs of bounded degree, graphs that satisfy certain covering conditions etc., somewhat better bounds are known [9, 8].

This leads to the following two natural questions concerning the rank of the 3D rigidity matroid. The first question was posed by Tibor Jordán during the 2008 BIRS rigidity workshop [1].

Question 1: Does *every* maximal, Maxwell-independent set of a graph have size at least the rank?

Question 2: Is there a better and tractable combinatorial rank upper bound (than the number of edges) for Maxwell-independent graphs?

Several algorithms exist for approximating the recognition of 3D rigidity and independence combinatorially ([15, 13, 16]). The simplest of these algorithms is a minor modification ([13]) of Jacobs and Hendrickson's ([10]) 2D pebble game, and finds a maximal Maxwell-independent set (it may be neither the minimum one nor the maximum one). Our answer to Question 1 simplifies the proof of correctness for these algorithms when they claim non-rigidity and non-independence of graphs in 3D.

Note that the answer to Question 1 would be obvious if every maximal Maxwell-independent set of a given graph G contains a maximal independent set of G . However, this is not the case. See Figure 2.

1.1 Contributions and organization

Our main results (Theorems 1, 3, 4 and 5) give affirmative answers to both questions. Theorem 1 is proved in Section 2 and gives affirmative answer to Question 1. Bill Jackson [6] has extended this result up to $d = 5$. His proof is

by contradiction and is hence nonconstructive. Our proof is constructive in the following sense: it uses Theorem 2(b) which elucidates the structure of a special type of cover that can always be found for Maxwell-independent but non-Maxwell-rigid graphs. This structure is of independent interest and is exploited (by Theorem 4) employing a commonly used rank inclusion-exclusion (IE) count. Our best bound (in Theorem 3) is employed in answering both Questions 1 (Theorem 1) and 2 (Theorems 4 and 5). For Maxwell-independent graphs, the proof (using Lemma 1) constructs a subgraph (independence assignment) whose size meets this bound, and moreover contains a maximal true independent set. The construction is not only of algorithmic interest, but also leads to bounds on rank for certain classes of non-Maxwell-independent graphs in Section 4.3.

In Section 4, we extend the same rank IE count bounds to more general covers that apply to Maxwell-independent and Maxwell-rigid graphs. This answers Question 2 in the affirmative. In Section 4.3 (Theorem 6), we additionally extend these to IE count bounds on Maxwell-dependent graphs that possess the appropriate type of covers. This yields a significantly simpler proof of correctness of an existing divide-and-conquer algorithm [15], that decomposes a Maxwell-independent graph recursively in order to upper bound the rank of the 3D rigidity matroid. In Section 4.1, we relate our bounds to existing bounds and conjectures. In the concluding Section 5, we pose open problems.

2 Main result and proof

In this section, we state and give the proof of the following main theorem, answering Question 1. Along the way we partially answer Question 2.

Theorem 1. *Let \mathcal{M} be a maximal Maxwell-independent set of a graph $G = (V, E)$ and \mathcal{I} be a maximal independent set in the 3-dimensional generic rigidity matroid of G . Then $|\mathcal{M}| \geq |\mathcal{I}|$.*

The proof requires a few definitions.

Definition 1. *The Maxwell count for a graph $G = (V, E)$ in d dimensions is $d|V| - |E|$. G is said to be Maxwell-rigid in d dimensions, if there exists a Maxwell-independent subset $E^* \subseteq E$ such that the Maxwell count of $G^* = (V, E^*)$ is at most $\binom{d+1}{2}$. As exceptions, j -cliques in d -dimensional space ($j \leq d-1$) are considered to be Maxwell-independent and Maxwell-rigid. A subgraph $G' = (V', E')$ induced by $V' \subseteq V$ is said to be a component of G , if it is Maxwell-rigid. In addition, G' is called a vertex-maximal component of G , if it is Maxwell-rigid and there is no superset of V' that also induces a Maxwell-rigid subgraph of G .*

Remark: “ d -sparse” and “ (k, l) -sparse” (in this case, $(d, \binom{d+1}{2})$ -sparse) have been used in the literature for Maxwell-independent graphs (see [6], [12]). But we note that dense and sparse graphs have a variety of different meanings in graph theory. Our terminology is motivated by Maxwell’s first observation in 1864 that every graph G that is rigid in d dimensions must contain a Maxwell-independent subgraph that has at least $d|V| - \binom{d+1}{2}$ edges.

Proof. (of Theorem 1) We give (i) the high level roadmap of the proof: the intuitive idea, new concepts and definitions, and (ii) the formal argument of how the theorem follows from various observations, lemmas and other theorems, whose statements and proofs are given in Section 3. See Figure 3 for a roadmap.

Note. We use \mathcal{M} to refer to both the set of edges \mathcal{M} over a vertex set V and the graph (V, \mathcal{M}) , when the vertex set V is clear from the context. In general, by $\text{rank}(G)$, we refer to the rank of the generic 3D rigidity matroid of G . In the remainder of Section 2 and in Section 3, the dimension d is fixed to be 3.

First, notice that if \mathcal{M} is itself independent, we are done. Similarly, if \mathcal{M} is Maxwell-rigid, we are done, since $|\mathcal{M}| = 3|V| - 6 \geq \text{rank}(G) = |\mathcal{I}|$.

In order to show $|\mathcal{M}| \geq |\mathcal{I}|$, we would like to first decompose \mathcal{M} into a *cover* by vertex-maximal components $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_n$.

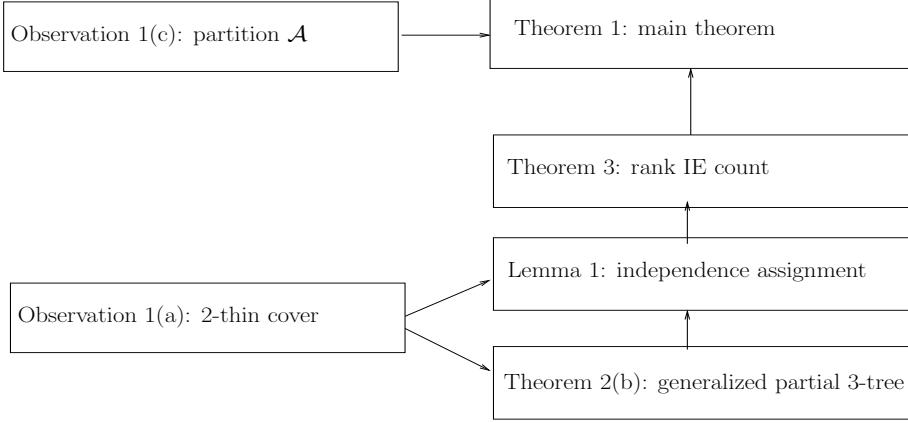


Figure 3: The roadmap for proof of Theorem 1.

Definition 2. A cover of a graph $G = (V, E)$ is a collection \mathcal{X} of pairwise incomparable subsets of V , each of size at least two, such that $\cup_{X \in \mathcal{X}} E(X) = E$. Alternatively, \mathcal{X} is a collection of induced subgraphs $(X, E(X))$. The rank inclusion-exclusion (IE) count of cover \mathcal{X} in 3D is defined as the following:

$$IE_{rank}(\mathcal{X}) := \sum_i^m rank(G_1(X_i)) - \sum_{(u,v) \in \mathcal{H}(\mathcal{X})} (n_{(u,v)} - 1)$$

Let $\mathcal{I}_{\mathcal{M}}$ with $|\mathcal{I}_{\mathcal{M}}| = \text{rank}(\mathcal{M})$ be a maximal independent set of \mathcal{M} . Without loss of generality, let $\mathcal{I}_{\mathcal{M}} \subseteq \mathcal{I}$. Let $\mathcal{A} := \mathcal{I} \setminus \mathcal{I}_{\mathcal{M}}$.

Consider a cover $\mathcal{X} = \{\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_n\}$ of \mathcal{M} by vertex-maximal components \mathcal{M}_i . A key observation proved in Observation 1(c) below is that for each (u, v) in \mathcal{A} , there exists at least one \mathcal{M}_i such that $u \in \mathcal{M}_i$ and $v \in \mathcal{M}_i$. Denote \mathcal{A} restricted to each \mathcal{M}_i by \mathcal{A}_i . Hence

$$|\mathcal{A}| \leq \sum_i |\mathcal{A}_i| \tag{1}$$

Observation 1. Let \mathcal{M} be a maximal Maxwell-independent set of a graph $G = (V, E)$. Let $\mathcal{X} = \{\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_n\}$ be any cover by vertex-maximal components of \mathcal{M} . Let $\mathcal{I}_{\mathcal{M}}$ be a maximal independent set of \mathcal{M} and extend $\mathcal{I}_{\mathcal{M}}$ to a maximal independent set \mathcal{I} of G . Let $\mathcal{A} = \mathcal{I} \setminus \mathcal{I}_{\mathcal{M}}$. Then

(c) If \mathcal{X} is the complete collection of vertex-maximal components of \mathcal{M} , and \mathcal{M} does not have Maxwell count 6, then for each edge $e = (u, v)$ in \mathcal{A} , there exists at least one \mathcal{M}_i such that $u \in \mathcal{M}_i$ and $v \in \mathcal{M}_i$. If we partition \mathcal{A} into $\cup_i \mathcal{A}_i$, where each \mathcal{A}_i is contained in an \mathcal{M}_i of \mathcal{M} , then we have

$$\sum_i |\mathcal{A}_i| \geq |\mathcal{A}| = |\mathcal{I} \setminus \mathcal{I}_{\mathcal{M}}|$$

Denote by $\mathcal{H}(\mathcal{X})$ the union of all edges e that lie in some intersection $\mathcal{M}_i \cap \mathcal{M}_j$ and for each such e , let n_e be the number of \mathcal{M}_i 's that contain e . It follows that

$$|\mathcal{M}| = \sum_i |\mathcal{M}_i| - \sum_{e \in \mathcal{H}(\mathcal{X})} (n_e - 1) \tag{2}$$

Since each \mathcal{M}_i is Maxwell-rigid, adding any $e \in \mathcal{A}_i$ into \mathcal{M}_i causes the number of edges in \mathcal{M}_i to exceed $3|V(\mathcal{M}_i)| - 6$ and in turn indicates the existence of a true dependence. However, since \mathcal{I} and $\mathcal{I}_{\mathcal{M}}$ are both independent, and $\mathcal{I}_{\mathcal{M}}$ spans \mathcal{M} , we know edges in \mathcal{A} are not in the span of edges in \mathcal{M} . I.e., the *true dependence should already have been in \mathcal{M}_i even before \mathcal{A}_i was added*. So we have

$$|\mathcal{M}_i| \geq \text{rank}(\mathcal{M}_i) + |\mathcal{A}_i| \tag{3}$$

Plugging (3) into (2), we have

$$|\mathcal{M}| \geq \sum_i \text{rank}(\mathcal{M}_i) - \sum_{e \in \mathcal{H}(\mathcal{X})} (n_e - 1) + \sum_i |\mathcal{A}_i| \quad (4)$$

Thus if we can prove

$$\sum_i \text{rank}(\mathcal{M}_i) - \sum_{e \in \mathcal{H}(\mathcal{X})} (n_e - 1) \geq \text{rank}(\mathcal{M}) = |\mathcal{I}_{\mathcal{M}}|, \quad (5)$$

then using (4) and (1), we can obtain that

$$\begin{aligned} |\mathcal{M}| &\geq |\mathcal{I}_{\mathcal{M}}| + \sum_i |\mathcal{A}_i| (\text{using (4) and (5)}) \\ &\geq |\mathcal{I}_{\mathcal{M}}| + |\mathcal{A}| (\text{using (1)}) \\ &= |\mathcal{I}|, \end{aligned}$$

which would prove Theorem 1. We will use Theorem 3 below to obtain (5) above. Theorem 3 needs the following definition of *independence assignment*.

Definition 3. Given a graph $G = (V, E)$ and a cover $\mathcal{X} = \{\mathcal{M}_1, \dots, \mathcal{M}_n\}$ of G , we say (G, \mathcal{X}) has an independence assignment $[\mathcal{I}; \{\mathcal{I}_1, \dots, \mathcal{I}_n\}]$, if there is an independent set \mathcal{I} of G and maximal independent set \mathcal{I}_i of each of the \mathcal{M}_i 's, such that \mathcal{I} restricted to \mathcal{M}_i , (denoted $\mathcal{I}|_i$), is contained in \mathcal{I}_i and for any $e \in \mathcal{H}(\mathcal{X})$, e is missing from at most one of the \mathcal{I}_i 's whose corresponding \mathcal{M}_i contains e . When \mathcal{X} is clear, we also say there is an independence assignment for G .

Theorem 3. If \mathcal{M} is a Maxwell-independent graph and $\mathcal{X} = \{\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_n\}$ is a cover of \mathcal{M} such that $(\mathcal{M}, \mathcal{X})$ has an independence assignment, then $\sum_i \text{rank}(\mathcal{M}_i) - \sum_{e \in \mathcal{H}(\mathcal{X})} (n_e - 1) \geq \text{rank}(\mathcal{M})$.

In order to apply Theorem 3 to prove the above (5), we use Lemma 1 below and Observation 1(a) at the beginning of Section 3. Lemma 1 states the existence of independence assignments for the so-called *2-thin covers* by components, which we first define below.

Definition 4. A cover $\mathcal{X} = \{X_1, X_2, \dots, X_m\}$ of G is 2-thin if $|X_i \cap X_j| \leq 2$ for all $1 \leq i < j \leq m$. Let $\mathcal{H}(\mathcal{X})$ be the set of all pairs of vertices (u, v) such that $X_i \cap X_j = \{u, v\}$ for some $1 \leq i < j \leq m$. The cover is strong 2-thin if $\mathcal{H}(\mathcal{X}) \subseteq E$. Denote by $n_{(u,v)}$ the number of elements in \mathcal{X} that contain both u and v and $G_1(X_i)$ as the subgraph of G induced by X_i with the $(u, v) \in \mathcal{H}(\mathcal{X})$ added as edges.

Remark: for strong 2-thin cover \mathcal{X} , each pair of vertices in $\mathcal{H}(\mathcal{X})$ is an edge e . Hence we have

$$IE_{\text{rank}}(\mathcal{X}) = \sum_i^m \text{rank}(G(X_i)) - \sum_{e \in \mathcal{H}(\mathcal{X})} (n_e - 1)$$

Now we can state Lemma 1.

Lemma 1. If \mathcal{M} is Maxwell-independent and \mathcal{X} is a 2-thin cover of \mathcal{M} by components of \mathcal{M} , then $(\mathcal{M}, \mathcal{X})$ has an independence assignment.

Lemma 1 in turn uses Theorem 2(b) below which requires the following definitions of *2-thin component graphs* and *generalized partial m-trees*.

Definition 5. Given graph $G = (V, E)$, let $\mathcal{X} = \mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_n$ be a 2-thin cover of G by components of G . The 2-thin component graph, or component graph for short, \mathcal{C}_G of G contains a component node for each component \mathcal{M}_i in \mathcal{C}_G and whenever \mathcal{M}_i and \mathcal{M}_j share an edge in G , their corresponding component nodes in \mathcal{C}_G are connected via an edge node.

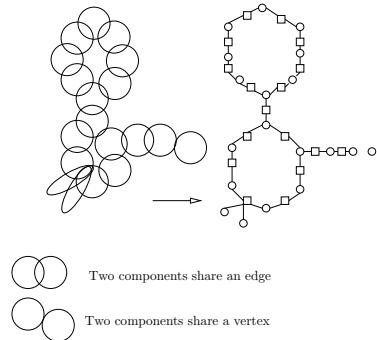


Figure 4: On the left side is a graph with its vertex-maximal components. On the right side is its 2-thin component graph, where circles represent component nodes and squares represent edge nodes. Note that the 2-thin component graph may not be connected.

For example, Figure 4 shows how to obtain a 2-thin component graph from a graph and its vertex-maximal components such that each component is a component node.

Note: components sharing only vertices are non-adjacent in the component graph. See Figure 4.

Theorem 2(b) below states that these 2-thin component graphs generalize the following concept of partial m -trees (also called *tree-width m graphs*) and Henneberg constructions [4].

Definition 6. For any $m \in \mathbb{N}$, we can define a generalized partial m -tree G as the following. All vertices of G that have degree at most m are leaves. Remove all leaves from G to get G_1 . The vertices with degree at most m in G_1 are the new leaves. From G_i , use the same process of removing leaves to get G_{i+1} . Continue this process until no vertex with degree at most m can be found. If the remaining graph \mathcal{K}_G , which is called a kernel graph, is empty, then G is a generalized partial m -tree.

Definition 7. For any $m \in \mathbb{N}$, all component nodes of a 2-thin component graph \mathcal{C}_G that have degree at most m are leaves. Remove all leaves from \mathcal{C}_G and all edge nodes that are of degree 1 after removing all the leaves are also removed. Denote the new 2-thin component graph as \mathcal{C}_{G_1} . The component nodes with degree at most m in \mathcal{C}_{G_1} are the new leaves. From \mathcal{C}_{G_i} , use the same process of removing leaves and degree 1 edge nodes to get $\mathcal{C}_{G_{i+1}}$. Continue this process until no vertex with degree less than $m + 1$ can be found. If the remaining graph $\mathcal{K}_{\mathcal{C}_G}$, which is called a kernel component graph, is empty, then we call \mathcal{C}_G a generalized partial m -tree.

Figure 5 gives an example of a generalized partial m -tree.

Now we can state Theorem 2(b).

Theorem 2(b). If \mathcal{M} is a Maxwell-independent graph and \mathcal{X} is a 2-thin cover of \mathcal{M} by components of \mathcal{M} , then any subgraph of the 2-thin component graph $\mathcal{C}_{\mathcal{M}}$ with respect to \mathcal{X} is a generalized partial 3-tree.

We can now conclude the proof of Theorem 1: from Observation 1(a) below at the beginning of Section 3, we know that the set of vertex-maximal components of \mathcal{M} forms a 2-thin cover. So we can apply Theorem 2(b), which permits us to construct the independence assignment for Lemma 1. Then, we can apply Theorem 3 to get (5) thus completing the proof of Theorem 1.

We summarize the proof below.

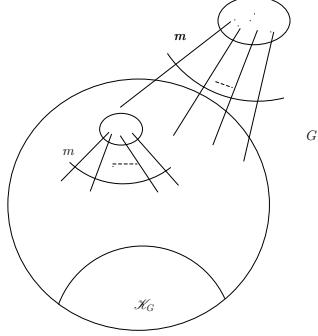


Figure 5: A generalized partial m -tree. In the figure, \mathcal{K}_G is the kernel graph. Each ellipses represents a component node. Component nodes with degree at most m are leaves.

Let $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_n$ be the set of all vertex-maximal components of \mathcal{M} . Let n_e be the number of \mathcal{M}_i 's that contain e . Let $\mathcal{A} = \mathcal{I} \setminus \mathcal{I}_{\mathcal{M}}$. Then

$$\begin{aligned} |\mathcal{M}| &= \sum_i |\mathcal{M}_i| - \sum_{e \in \mathcal{H}(\mathcal{X})} (n_e - 1) \quad \text{from (2)} \\ &\geq \sum_i (|\mathcal{A}_i| + \text{rank}(\mathcal{M}_i)) - \sum_{e \in \mathcal{H}(\mathcal{X})} (n_e - 1) \quad \text{from (3)} \\ &\geq |\mathcal{A}| + \sum_i \text{rank}(\mathcal{M}_i) - \sum_{e \in \mathcal{H}(\mathcal{X})} (n_e - 1) \quad \text{from Observation 1(c)} \end{aligned}$$

Using Theorem 3 which applies Observation 1 (a) and Lemma 1 (which in turn uses Theorem 2(b)), we have:

$$\begin{aligned} |\mathcal{A}| + \sum_i \text{rank}(\mathcal{M}_i) - \sum_{e \in \mathcal{H}(\mathcal{X})} (n_e - 1) &\geq |\mathcal{A}| + \text{rank}(\mathcal{M}) \quad \text{by Theorem 3} \\ &\geq |\mathcal{I}| = \text{rank}(G). \end{aligned}$$

□

3 Proofs of intermediate results

In the following we fill in the pieces of the above proof. We prove Observation 1, Theorem 2, Lemma 1 and Theorem 3.

Observation 1. Let \mathcal{M} be a maximal Maxwell-independent set of a graph $G = (V, E)$. Let $\mathcal{X} = \{\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_n\}$ be any cover by vertex-maximal components of \mathcal{M} . Let $\mathcal{I}_{\mathcal{M}}$ be a maximal independent set of \mathcal{M} and extend $\mathcal{I}_{\mathcal{M}}$ to a maximal independent set \mathcal{I} of G . Let $\mathcal{A} = \mathcal{I} \setminus \mathcal{I}_{\mathcal{M}}$. Then

(a) \mathcal{X} forms a 2-thin cover of \mathcal{M} and

$$|\mathcal{M}| = \sum_i |\mathcal{M}_i| - \sum_{e \in \mathcal{H}(\mathcal{X}_{\mathcal{M}})} (n_e - 1)$$

(b) \mathcal{X} forms a strong 2-thin cover of \mathcal{M} .

(c) If \mathcal{X} is the complete collection of vertex-maximal components of \mathcal{M} and \mathcal{M} does not have Maxwell count 6, then for each edge $e = (u, v)$ in \mathcal{A} , there exists at least one \mathcal{M}_i such that $u \in \mathcal{M}_i$ and $v \in \mathcal{M}_i$. If we partition \mathcal{A} into $\cup_i \mathcal{A}_i$, where each \mathcal{A}_i is contained in an \mathcal{M}_i of \mathcal{M} , then we have

$$\sum_i |\mathcal{A}_i| \geq |\mathcal{A}| = |\mathcal{I} \setminus \mathcal{I}_{\mathcal{M}}|$$

Proof. (a) Suppose \mathcal{M}_j and \mathcal{M}_k are two vertex-maximal components in \mathcal{M} . If \mathcal{M}_j and \mathcal{M}_k share more than 2 vertices, then their union will be Maxwell-rigid, since \mathcal{M} is Maxwell-independent and $\mathcal{M}_j \cup \mathcal{M}_k$ will have Maxwell count at most 6.

(b) Suppose \mathcal{M}_j and \mathcal{M}_k are two vertex-maximal components in \mathcal{M} . If they \mathcal{M}_j and \mathcal{M}_k share two vertices but do not share an edge, then their union will also be Maxwell-rigid, which violates the vertex-maximal property of \mathcal{M}_j and \mathcal{M}_k .

(c) Assume there is an edge $e = (u, v) \in \mathcal{A}$ such that u and v are not both inside any vertex-maximal components of \mathcal{M} . Then e itself becomes a component of $\mathcal{M} \cup \{e\}$. Then $\mathcal{M} \cup \{e\}$ is also Maxwell-independent, since \mathcal{M} does not have Maxwell count 6. This contradicts to the fact that \mathcal{M} is maximal Maxwell-independent of G . \square

Remark: The full strength of Theorems 4, 5, 6, 7 and equality in (1) all rely on Observation 1(b) (strongness of 2-thin cover). This is not required to prove Theorem 1. Observation 1(a) (2-thin cover) is sufficient for Theorem 1.

For a Maxwell-independent graph \mathcal{M} and a 2-thin cover \mathcal{X} of \mathcal{M} , the 2-thin component graph $\mathcal{C}_{\mathcal{M}}$ has the following property:

Theorem 2. *If \mathcal{M} is a Maxwell-independent graph and \mathcal{X} is a 2-thin cover of \mathcal{M} by components of \mathcal{M} , then*

(a) *any subgraph of the 2-thin component graph $\mathcal{C}_{\mathcal{M}}$ with respect to \mathcal{X} has average degree strictly less than 4.*
 (b) *any subgraph of the 2-thin component graph of \mathcal{M} with respect to \mathcal{X} is a generalized partial 3-tree.*

Proof. (a) Let $\mathcal{K}_{\mathcal{M}}$ be any subgraph of the 2-thin component graph $\mathcal{C}_{\mathcal{M}}$ with respect to \mathcal{X} . Let K denote $\mathcal{K}_{\mathcal{M}}$'s corresponding subgraph in \mathcal{M} . Let $\mathcal{X}_K := \{\mathcal{M}_1, \dots, \mathcal{M}_n\}$ be \mathcal{X} restricted to K . Let V_i and E_i be the shared vertex and shared edge sets of component node \mathcal{M}_i of K that are shared by other component nodes \mathcal{M}_j of K . Let V_s and E_s be the entire sets of such shared vertices and shared edges in K . Denote the average degree of vertices in the shared graph as $s = 2 \frac{|E_s|}{|V_s|}$ and the average degree of a shared vertex within a component node as $w = 2 \frac{\sum_i |E_i|}{\sum_i |V_i|}$. Let n_e and n_v denote the number of component nodes \mathcal{M}_i of K that share e and v respectively. Since the Maxwell count of each \mathcal{M}_i is 6, the Maxwell count of K can be calculated as follows:

$$\sum_i 6 - 3 \sum_{v \in V_s} n_v + \sum_{e \in E_s} n_e + 3|V_s| - |E_s| = \sum_i (6 - 3|V_i| + |E_i|) + 3|V_s| - |E_s|$$

Suppose the Maxwell count of K is ≥ 6 . We have

$$6n - 6 \geq 3 \sum_i |V_i| - \sum_i |E_i| - 3|V_s| + |E_s| \quad (6)$$

Consider any shared vertex v in V_s . Denote by $C_v \subseteq \{1, \dots, n\}$ the set of indices of component nodes that are incident at v . In this proof, since the context is clear, we refer to \mathcal{M}_j , $j \in C_v$ as a component node incident at v . The collection of all n_v component nodes of K meeting at v forms a subgraph C . Since K is Maxwell-independent, C is also Maxwell-independent. Let w_v^j be the number of shared edges incident at v in component node \mathcal{M}_j and s_v be the number of shared edges that are incident at v . Then the Maxwell count of C can be computed as follows:

- there are n_v component nodes, which contributes $6n_v$;

- v is shared by n_v component nodes, and the contribution is $-(3n_v - 3)$;
- each shared edge in a component node \mathcal{M}_j contributes 1 to the Maxwell count, and in total the shared edges contribute $(\sum_{j \in C_v} w_v^j) - s_v$
- for each shared edge $e = (u, v)$, vertex u contributes $-3[(\sum_{j \in C_v} w_v^j) - s_v]$

Thus the Maxwell count of C is:

$$3n_v - 2[(\sum_{j \in C_v} w_v^j) - s_v] + 3$$

Since C is Maxwell-independent, we know:

$$\begin{aligned} 3n_v - 2[(\sum_{j \in C_v} w_v^j) - s_v] + 3 &\geq 6 \\ 3n_v - 2[(\sum_{j \in C_v} w_v^j) - s_v] &\geq 3 \end{aligned}$$

Summing over all shared vertices in V_s , we have:

$$3 \sum_{v \in V_s} n_v - 2 \sum_{v \in V_s} [(\sum_{j \in C_v} w_v^j) - s_v] \geq 3|V_s|$$

Since $\sum_{v \in V_s} n_v = \sum_i |V_i|$, $\sum_{v \in V_s} (\sum_{j \in C_v} w_v^j) = 2 \sum_i |E_i|$ and $\sum_{v \in V_s} s_v = 2|E_s|$, we know

$$3 \sum_i |V_i| - 4 \sum_i |E_i| - 3|V_s| + 4|E_s| \geq 0$$

Plugging into (6), we have:

$$\begin{aligned} 6n - 6 &\geq 3 \sum_i |V_i| - \sum_i |E_i| - 3|V_s| + |E_s| \\ &\geq 3 \sum_i |V_i| - 4 \sum_i |E_i| - 3|V_s| + 4|E_s| \\ &\quad + 3(\sum_i |E_i| - |E_s|) \\ &\geq 3(\sum_i |E_i| - |E_s|) \end{aligned}$$

Since $|E_s| \leq \frac{1}{2} \sum_i |E_i|$, we have:

$$6n - 6 \geq \frac{3}{2} \sum_i |E_i|$$

Assume in K , the component nodes have average degree at least 4, i.e. $\sum_i |E_i| \geq 4n$. Hence we have:

$$6n - 6 \geq \frac{3}{2} 4n = 6n.$$

Contradiction. Thus the average degree of the component nodes in K is strictly less than 4.

(b) Assume the contrary: there exists a subgraph S of the 2-thin component graph of \mathcal{M} with respect to \mathcal{X} that is not a generalized partial 3-tree. Then the kernel component graph \mathcal{K}_S is not empty. I.e, in \mathcal{K}_S , every component node has degree at least 4 hence the average degree is greater than 4. But \mathcal{K}_S is a subgraph of the 2-thin component graph of \mathcal{M} with respect to \mathcal{X} and from Theorem 2(a), we know the average degree of the component nodes in \mathcal{K}_S is strictly less than 4. Contradiction. \square

Remark: Theorem 2(a) shows that there is no subgraph of the 2-thin component graph where each component node has at least 4 shared edges. It is tempting to try to use the counts for the so-called “identified” body-hinge frameworks here [18, 20, 11, 17], treating the component nodes as bodies and the shared edges as hinges. However, while identified body-hinge frameworks account for *several* component nodes sharing an edge (as we have here), we additionally have shared edges that have common vertices, hence the generic, identified body-hinge counts may not apply.

Next we show the existence of an independence assignment for Maxwell-independent graphs.

Lemma 1. *If \mathcal{M} is Maxwell-independent and \mathcal{X} is a 2-thin cover of \mathcal{M} by components of \mathcal{M} , then $(\mathcal{M}, \mathcal{X})$ has an independence assignment.*

Proof. (of Lemma 1). In fact, if the 2-thin component graph of \mathcal{M} is a generalized partial 9-tree, we can construct an independence assignment. From Theorem 2(b), we know that any subgraph of the 2-thin component graph $\mathcal{C}_{\mathcal{M}}$ of \mathcal{M} is a generalized partial 3-tree, which is automatically a generalized partial 9-tree. Let $\mathcal{M}_1, \mathcal{M}_2, \dots$ be the component nodes of \mathcal{M} listed in reverse order from the removal order in Definition 6. We use induction to prove that there is always an independence assignment for $(\mathcal{M}, \mathcal{X})$.

If \mathcal{M} itself is Maxwell-rigid, it is clear that we can find an independence assignment.

Suppose there is an independence assignment $[\emptyset \mathcal{I}^k; \mathcal{I}_i^k \mid 1 \leq i \leq k]$ for a subgraph $\mathcal{C}_{\mathcal{M}}^k$ of $\mathcal{C}_{\mathcal{M}}$ containing the component nodes $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_k$. When we add \mathcal{M}_{k+1} to form $\mathcal{C}_{\mathcal{M}}^{k+1}$, we can keep the edges of \mathcal{I}^k in \mathcal{I}^{k+1} noticing that \mathcal{S} , the set of shared edges of \mathcal{M}_{k+1} is in the span of \mathcal{I}^k . To get the new maximal independent set, first we take $\mathcal{I}_i^{k+1} := \mathcal{I}_i^k$ for $1 \leq i \leq k$. Then we find a maximal independent set \mathcal{I}_{k+1}^{k+1} within \mathcal{M}_{k+1} that contains all its shared edges \mathcal{S} . Since $|\mathcal{S}| \leq 9$, \mathcal{S} is independent in $3D$, because in $3D$, a minimum-size graph that is not independent will have at least 10 edges. If $\mathcal{I}^k \cup (\mathcal{I}_{k+1}^{k+1} - \mathcal{S})$ is independent, then we have found a new assignment. Otherwise, to get \mathcal{I}^{k+1} , we remove edges in $\mathcal{I}_{k+1}^{k+1} - \mathcal{S}$ until its union with \mathcal{I}^k is independent. Note that in the process of removal, all edges of the generalized partial 9-tree are always in the span of the \mathcal{I}^{k+1} . Hence we can find an assignment for the new generalized partial 9-tree $\mathcal{C}_{\mathcal{M}}^{k+1}$. \square

With an independence assignment, we can prove the following theorem, *which partially answers Question 2 in the affirmative (for Maxwell-independent but not Maxwell-rigid graphs).*

Theorem 3. *If \mathcal{M} is a Maxwell-independent graph and \mathcal{X} is a cover $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_n$ of \mathcal{M} such that $(\mathcal{M}, \mathcal{X})$ has an independence assignment, then $\sum_i \text{rank}(\mathcal{M}_i) - \sum_{e \in \mathcal{H}(\mathcal{X})} (n_e - 1) \geq \text{rank}(\mathcal{M})$.*

Proof. Let $[\mathcal{I}_{\mathcal{M}}; \{\mathcal{I}_{\mathcal{M}_1}, \dots, \mathcal{I}_{\mathcal{M}_n}\}]$ be the independence assignment. Thus $\mathcal{I}_{\mathcal{M}}|_i \subseteq \mathcal{I}_{\mathcal{M}_i}$ and whenever $e \in \mathcal{H}(\mathcal{X})$, it holds that $e \in \mathcal{I}_{\mathcal{M}_i}$ for at least $n_e - 1$ of the \mathcal{M}_i 's sharing e .

$$\sum_i \text{rank}(\mathcal{M}_i) - \sum_{e \in \mathcal{H}(\mathcal{X})} (n_e - 1) = \sum_i |\mathcal{I}_{\mathcal{M}_i}| - \sum_{e \in \mathcal{H}(\mathcal{X})} (n_e - 1)$$

Notice that every e in $\mathcal{I}_{\mathcal{M}}$ contributes exactly 1 to the above sum (this is true by inclusion-exclusion regardless whether $e \in \mathcal{H}(\mathcal{X})$ or not because $\mathcal{I}_{\mathcal{M}}|_i \subseteq \mathcal{I}_{\mathcal{M}_i}$). Furthermore, any $e \notin \mathcal{I}_{\mathcal{M}}$ contributes a non-negative number (1 or 0) to the above sum. (In fact, only $e \in \bigcup_i \mathcal{I}_{\mathcal{M}_i}$ contributes anything). So we have the following:

$$\begin{aligned} & \sum_i |\mathcal{I}_{\mathcal{M}_i}| - \sum_{e \in \mathcal{H}(\mathcal{X})} (n_e - 1) \\ & \geq |\mathcal{I}_{\mathcal{M}}| \\ & = \text{rank}(\mathcal{M}) \end{aligned}$$

\square

4 Better bounds using IE counts

4.1 Relation to known bounds and conjectures using IE counts

Decomposition of graphs into covers is a natural way of approaching a characterization of 3D rigidity matroid. So far, the inclusion-exclusion(IE) count method for covers has been used by many in the literature, such as [2][16][13][15][14] [8] and [9]. The most explored decompositions are the 2-thin covers.

In 1983, Dress et al [3, 19] conjectured that the minimum of the rank IE count taken over all 2-thin covers is an upper bound on the rank of 3D rigidity matroid. However, this conjecture was disproved for general graphs by Jackson and Jordán in [7]. In fact, in [9] they show that the minimum of the rank IE count taken over all *independent* 2-thin covers is an upper bound on the rank. Here, an *independent 2-thin cover* \mathcal{X} is one for which the edge set given by the pairs in $\mathcal{H}(\mathcal{X})$ is independent. Recall that $\mathcal{H}(\mathcal{X})$ was defined in Definition 2.

Our Theorem 4, and Theorem 5 below in Section 4.2, will show that the same rank IE count over a *specific, not necessarily independent* cover gives a rank upper bound for Maxwell-independent graphs. We have no examples where our bound is better than [9]. In fact, they conjecture that their bound is tight when restricted to non-rigid graphs and covers of size at least 2, hence Bill Jackson pointed out that any such examples would be counterexamples to their conjecture. However, our bound may be more tractable, since we use a specific, not necessarily independent cover by (proper) vertex-maximal components, thereby answering Question 2 in the Introduction.

Besides the rank IE count, other IE counts have also been explored in the aforementioned literature.

Definition 8. Given graph G and a cover $\mathcal{X} = \{X_1, X_2, \dots, X_m\}$ of G . The full rank inclusion-exclusion of cover \mathcal{X} in 3D is defined as $IE_{full}(\mathcal{X}) := \sum_i^m (3|V_i| - 6) - \sum_{(u,v) \in \mathcal{H}(\mathcal{X})} (n_{(u,v)} - 1)$, where V_i is the vertex set of X_i and $\mathcal{H}(\mathcal{X})$ and $n_{(u,v)}$ are as in Definition 2.

In [9], Jackson and Jordán defined covers that need not be independent, but are obtained as iterated, or recursive version of independent covers and showed that the minimum IE_{full} count taken over all iterated 2-thin covers is an upper bound on rank.

In Section 4.3, we use the same IE_{full} count over a *specific, non-iterated, non-independent* cover, to obtain rank bounds on Maxwell-dependent graphs. Again, we have no examples where our bound is better than [9]. In fact they additionally conjectured that their bound is tight, hence Bill Jackson pointed out that any such examples would be counterexamples to their conjecture. Our bound may be more tractable, since we use a specific, non-iterated, not necessarily independent cover by (proper) vertex-maximal, components. However, the catch is that these covers may not exist for general graphs.

4.2 Better bounds for Maxwell-independent graphs

Next we answer Question 2 in the Introduction in the affirmative.

When a graph G is not Maxwell-rigid, Theorem 3 gives the required answer. In fact, we can observe that there is a more general form of the inclusion-exclusion expression of Theorem 3 for Maxwell-independent graphs, using the fact that the cover is strong 2-thin.

Theorem 4. Given a Maxwell-independent graph \mathcal{M} and any set \mathcal{X} of vertex-maximal components $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_n$ that is a cover of \mathcal{M} , then the rank IE count of cover \mathcal{X} is at least $rank(\mathcal{M})$, i.e, $\sum_i rank(\mathcal{M}_i) - \sum_{(u,v) \in \mathcal{H}(\mathcal{X})} (n_{(u,v)} - 1) \geq rank(\mathcal{M})$.

Proof. From Observation 1(a), we know \mathcal{X} is 2-thin. So we can apply Lemma 1 and obtain that there is an independence assignment of \mathcal{I} for \mathcal{M} and \mathcal{I}_i 's for the components in \mathcal{X} . Since Observation 1(b) shows that \mathcal{X} is strong 2-thin, we can replace $(u, v) \in \mathcal{H}(\mathcal{X})$ by $e \in \mathcal{H}(\mathcal{X})$ everywhere. Let n_e be the number of \mathcal{M}_i 's that contain e . Then for each $e \in \mathcal{H}(\mathcal{X}) - \mathcal{I}$, e appears in at least $(n_e - 1)$ \mathcal{I}_i 's whose corresponding components contain e .

Applying Theorem 3, we have $\sum_i \text{rank}(\mathcal{M}_i) - \sum_{e \in \mathcal{H}(\mathcal{X})} (n_e - 1) \geq \text{rank}(\mathcal{M})$. In other words,

$$\begin{aligned} IE_{\text{rank}} &= \sum_i \text{rank}(\mathcal{M}_i) - \sum_{(u,v) \in \mathcal{H}(\mathcal{X})} (n_{(u,v)} - 1) \\ &= \sum_i \text{rank}(\mathcal{M}_i) - \sum_{e \in \mathcal{H}(\mathcal{X})} (n_e - 1) \\ &\geq \text{rank}(\mathcal{M}) \end{aligned}$$

□

Remark: Theorem 3 does not need the cover to be strong 2-thin, or even 2-thin, and is sufficient in proving Theorem 1. Theorem 4, on the other hand, needs the cover to be strong 2-thin.

When G is Maxwell-rigid, there is a single vertex-maximal component namely G itself, so the bound of Theorem 3 is uninteresting. In this case, we use the cover of G by “proper” vertex-maximal components:

Definition 9. Given graph $G = (V, E)$, an induced subgraph is proper vertex-maximal, Maxwell-rigid if it is Maxwell-rigid and the only graph that properly contains this subgraph and is Maxwell-rigid is G itself.

We note that the collection of proper vertex-maximal components may not be 2-thin cover even for Maxwell-independent graphs.

Theorem 5. Given a Maxwell-independent graph (V, \mathcal{M}) and any set of proper vertex-maximal components $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_n$ that forms a cover \mathcal{X} , then the rank IE count of \mathcal{X} , i.e., $\sum_i \text{rank}(\mathcal{M}_i) - \sum_{(u,v) \in \mathcal{H}(\mathcal{X})} (n_{(u,v)} - 1)$ is at least $\text{rank}(\mathcal{M})$.

Proof. If \mathcal{M} is not Maxwell-rigid, then $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_n$ will be vertex-maximal components. Then we can apply Theorem 4.

From Observation 1(a) and Observation 1(b), we know that the cover is automatically strong 2-thin and there is an independence assignment of \mathcal{I} and \mathcal{I}_i in the cover. Rank IE count of \mathcal{X} is then at least $\text{rank}(\mathcal{M})$.

If \mathcal{M} is Maxwell-rigid, we have the following cases:

Case 1 The cover is not strong 2-thin. This can be further divided into two cases.

Case 1a. The cover is not 2-thin: there exist \mathcal{M}_i and \mathcal{M}_j such that their intersection has at least 3 vertices.

From Observation 1(a) we know the union of \mathcal{M}_i and \mathcal{M}_j is also Maxwell-rigid and thus $\mathcal{M} = \mathcal{M}_i \cup \mathcal{M}_j$. We can start from a maximal independent set $\mathcal{I}_{\mathcal{H}(\mathcal{X})}$ of $\mathcal{M}_i \cap \mathcal{M}_j$, and expand it to find the maximal independent sets \mathcal{I}_i of \mathcal{M}_i and \mathcal{I}_j of \mathcal{M}_j . It is clear that $\mathcal{I}_i \cup \mathcal{I}_j$ spans the whole graph \mathcal{M} , and $IE_{\text{rank}} = \text{rank}(\mathcal{M}_i) + \text{rank}(\mathcal{M}_j) - \text{rank}(\mathcal{M}_i \cap \mathcal{M}_j) = |\mathcal{I}_i \cup \mathcal{I}_j| \geq |\mathcal{I}| = \text{rank}(\mathcal{M})$.

Case 1b. The cover is 2-thin but not strong 2-thin. Then there are exactly 2 components \mathcal{M}_1 and \mathcal{M}_2 , since if there are 3 or more components, two components whose intersection is a pair of nonadjacent vertices would form a component, violating proper-maximality, unless the third component is a single edge. But if the third component is a single edge, the union of the three components violates Maxwell-independence of the whole graph.

Denote by $\mathcal{I}_{\mathcal{M}}$ a maximal independent set of \mathcal{M} . Let $\mathcal{I}_{\mathcal{M}_1}$ be a maximal independent set of \mathcal{M}_1 and $\mathcal{I}_{\mathcal{M}_2}$ be a maximal independent set of \mathcal{M}_2 . Recall that $(\mathcal{I}_{\mathcal{M}})|_{\mathcal{M}_1}$ means $\mathcal{I}_{\mathcal{M}}$ restricted to \mathcal{M}_1 . Then $(\mathcal{I}_{\mathcal{M}})|_{\mathcal{M}_1} \subseteq \mathcal{M}_1$ and $(\mathcal{I}_{\mathcal{M}})|_{\mathcal{M}_2} \subseteq \mathcal{M}_2$. Thus

$$\begin{aligned} IE_{\text{rank}} &= |\mathcal{I}_{\mathcal{M}_1} \cup \mathcal{I}_{\mathcal{M}_2}| \\ &\geq |(\mathcal{I}_{\mathcal{M}})|_{\mathcal{M}_1} \cup (\mathcal{I}_{\mathcal{M}})|_{\mathcal{M}_2}| \\ &= |\mathcal{I}_{\mathcal{M}}| \\ &= \text{rank}(\mathcal{M}) \end{aligned}$$

Case 2. The cover is strong 2-thin. Then Theorem 2(b) (only needs 2-thin cover), Lemma 1 (only needs 2-thin cover) and Theorem 4 (needs strong 2-thin cover) apply.

□

4.3 Removing the Maxwell-independence condition

We have the following theorem for general graphs that may not be Maxwell-independent.

Theorem 6. *Given graph $G = (V, E)$, if the complete collection $\mathcal{X} = \{C_1, C_2, \dots, C_m\}$ of vertex-maximal components forms a 2-thin cover, then the IE_{full} count of the cover \mathcal{X} is an upper bound on $\text{rank}(G)$, i.e.,*

$$\sum_i^m (3|V_i| - 6) - \sum_{(u,v) \in H(\mathcal{X})} (n_{(u,v)} - 1) \geq \text{rank}(G),$$

where $V_i = V(C_i)$ for $1 \leq i \leq m$.

Proof. We first show that the cover is strong 2-thin. Suppose not, then there exists $(u, v) \in H(\mathcal{X})$ such that $(u, v) \notin E$. Suppose further that C_i and C_j both contain u and v . There exist Maxwell-independent sets \mathcal{M}_i of C_i and \mathcal{M}_j of C_j such that $|\mathcal{M}_i| = 3|V_i| - 6$ and $|\mathcal{M}_j| = 3|V_j| - 6$. We will show $\mathcal{M}_i \cup \mathcal{M}_j$ is also Maxwell-independent thus $(V_i \cup V_j, \mathcal{M}_i \cup \mathcal{M}_j)$ is Maxwell-rigid, a violation to the fact that C_i and C_j are vertex-maximal, Maxwell-rigid. Suppose $\mathcal{M}_i \cup \mathcal{M}_j$ is not Maxwell-independent, then there exists $\mathcal{M}' \subseteq \mathcal{M}_i \cup \mathcal{M}_j$ such that the subgraph induced by \mathcal{M}' has Maxwell count less than 6. It is clear that $\mathcal{M}' \not\subseteq \mathcal{M}_i$ and $\mathcal{M}' \not\subseteq \mathcal{M}_j$. Let $\mathcal{M}' = \mathcal{M}'_i \cup \mathcal{M}'_j$ such that $\mathcal{M}'_i \subseteq \mathcal{M}_i$ and $\mathcal{M}'_j \subseteq \mathcal{M}_j$. Since $(u, v) \notin E$, it can be seen that in order to make the subgraph induced by \mathcal{M}' have Maxwell count less than 6, one of the subgraphs G'_i induced by \mathcal{M}'_i and G'_j induced by \mathcal{M}'_j will have Maxwell count less than 6, which together with the fact that $\mathcal{M}'_i \subseteq \mathcal{M}_i$ and $\mathcal{M}'_j \subseteq \mathcal{M}_j$ violates Maxwell-independence of \mathcal{M}_i and \mathcal{M}_j . Hence the cover \mathcal{X} is strong 2-thin.

Next we construct an edge set \mathcal{M} whose size is equal to the IE_{full} count of cover \mathcal{X} . For $1 \leq i \leq m$, denote by \mathcal{M}_i a maximum-size Maxwell-independent set of C_i . Then for any $e \in H(\mathcal{X})$, e is not present in only one of the \mathcal{M}_i 's that contain both endpoints of e . Otherwise, if e is Maxwell-dependent on both \mathcal{M}_i and \mathcal{M}_j ($i \neq j$), i.e., e does not appear in either \mathcal{M}_i or \mathcal{M}_j but \mathcal{M}_i and \mathcal{M}_j share the two endpoints of e , then $\mathcal{M}_i \cup \mathcal{M}_j$ is also Maxwell-independent since $\mathcal{M}_i \cup \mathcal{M}_j$ has Maxwell count 6. That means $C_i \cup C_j$ is Maxwell-rigid, which is a violation to the vertex-maximality of C_i and C_j .

Thus, as shown in Figure 6 edges restricted to component C_i can be divided into four parts:

- \mathcal{P}_1^i : the set of edges e in $H(\mathcal{X}) \cap \mathcal{M}_i$ that are present in each \mathcal{M}_j for which C_j contains e ;
- \mathcal{P}_2^i : the set of edges e in $H(\mathcal{X}) \cap \mathcal{M}_i$ that are Maxwell-dependent on exactly one \mathcal{M}_j where C_j contains e ;
- \mathcal{P}_3^i : the set of edges e in $H(\mathcal{X}) \setminus \mathcal{M}_i$, and present in all other \mathcal{M}_j 's, where C_j contains e .
- \mathcal{P}_4^i : $C_i \setminus H(\mathcal{X})$.

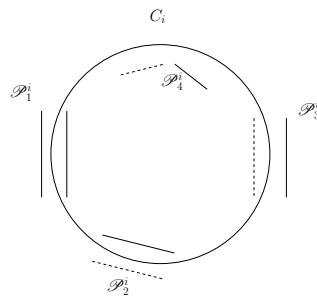


Figure 6: Edges in C_i are divided into four parts: the solid lines represent the edges that are in \mathcal{M}_i , and the dashed lines represent edges that are not in \mathcal{M}_i but in C_i .

Let $\mathcal{P}_k = \bigcup_i \mathcal{P}_k^i$. Now we construct the edge set \mathcal{M} by removing all edges in \mathcal{P}_2 and \mathcal{P}_3 from $\bigcup_i \mathcal{M}_i$. Now note that $|\mathcal{M}| = \sum_i^m (3|V_i| - 6) - \sum_{e \in \mathcal{P}_1} (n_e - 1) - \sum_{e \in \mathcal{P}_2 \cup \mathcal{P}_3} (n_e - 1)$, which is exactly the IE_{full} count of cover \mathcal{X} . In the following we show that \mathcal{M} is maximal Maxwell-independent:

(I) \mathcal{M} is Maxwell-independent. Suppose not, then we can find a minimal edge set $\mathcal{M}' \subseteq \mathcal{M}$ that is Maxwell-dependent. Since \mathcal{M} is picked in such a way that every $\mathcal{M}|_i$ is Maxwell-independent, we know \mathcal{M}' can not be inside any C_i . Because \mathcal{M}' is minimal, we know there exists $\mathcal{M}'' \subset \mathcal{M}'$ that is Maxwell-independent with Maxwell count 6. Denote by V' the vertex set of \mathcal{M}' . Then (V', \mathcal{M}'') is a component that is not contained in any C_i , since \mathcal{M}' is not inside any C_i , and removing an edge from \mathcal{M}' does not make it inside any C_i either. That is a contradiction to the fact that C_1, \dots, C_m is the complete collection of vertex-maximal components of G .

(II) \mathcal{M} is maximal Maxwell-independent. In order to show this, we first note that $e \in \mathcal{P}_2^i$ is Maxwell-independent in all non-trivial Maxwell-independent sets of C_i : suppose there exists some maximal Maxwell-independent set \mathcal{M}'_i of C_i such that e is Maxwell-dependent in \mathcal{M}'_i . Then there must be a subset \mathcal{M}''_i of \mathcal{M}'_i such that \mathcal{M}''_i has Maxwell count 6. Then for a component C_j that shares e with C_i , we know $\mathcal{M}''_i \cup \mathcal{M}_j$ also has Maxwell count 6 thus a contradiction to the vertex-maximality of C_j .

Suppose there is an edge $e \notin \mathcal{M}$ such that for every Maxwell-independent subgraph $\mathcal{M}_e \subseteq \mathcal{M}$, $\mathcal{M}_e \cup \{e\}$ is Maxwell-independent. Hence $(\mathcal{M} \cup e)|_i$ is Maxwell-independent. Recall that $(\mathcal{M} \cup e)|_i$ denotes $\mathcal{M} \cup e$ restricted to C_i . Since $e \in C_i$ for some i , due to completeness of the cover \mathcal{X} , we know $e \in \mathcal{P}_2^i, \mathcal{P}_3^i$ or \mathcal{P}_4^i . In fact every \mathcal{P}_2^j edge is a \mathcal{P}_3^i edge for some i , without loss of generality, we know $e \in \mathcal{P}_3^i$ or \mathcal{P}_4^i . Hence there is an extension of $(\mathcal{M} \cup e)|_i$ into a maximal Maxwell-independent set of C_i , which must contain all edges in \mathcal{P}_2^i as shown in the previous paragraph. This extension has size larger than \mathcal{M}_i , which is a contradiction to the fact that \mathcal{M}_i is a maximum Maxwell-independent set of C_i . Hence \mathcal{M} is maximal Maxwell-independent.

Thus we know \mathcal{M} is a maximal Maxwell-independent set of G . From Theorem 1, we know $|\mathcal{M}| \geq \text{rank}(G)$. As noticed before, the IE_{full} count of the cover \mathcal{X} is equal to $|\mathcal{M}|$, hence we have $\sum_i^m (3|V_i| - 6) - \sum_{e \in \mathcal{H}(\mathcal{X})} (n_e - 1) \geq \text{rank}(G)$. \square

Theorem 7. *Given graph $G = (V, E)$, if the complete collection $\mathcal{X} = \{C_1, C_2, \dots, C_m\}$ of proper vertex-maximal components forms a 2-thin cover, then the IE_{full} count of the cover \mathcal{X} is an upper bound on $\text{rank}(G)$, i.e.,*

$$\sum_i^m (3|V_i| - 6) - \sum_{(u,v) \in \mathcal{H}(\mathcal{X})} (n_{(u,v)} - 1) \geq \text{rank}(G),$$

where $V_i = V(C_i)$ for $1 \leq i \leq m$.

Proof. First, as the proof of Theorem 6, we can show that \mathcal{X} is strong 2-thin.

When G is not Maxwell-rigid, the remainder of the proof is the same as in Theorem 6.

When G is Maxwell-rigid, the major part of the proof is the same as in Theorem 6. However, since \mathcal{X} is a complete collection of proper vertex-maximal components, we can no longer use vertex-maximality to show that $e \in \mathcal{H}(\mathcal{X})$ is Maxwell-dependent in only one of the \mathcal{M}_i 's that contain both endpoints of e . Nevertheless, with proper vertex-maximality, if $e \in \mathcal{H}(\mathcal{X})$ is Maxwell-dependent in more than one of the \mathcal{M}_i 's that contain both endpoints of e , we know that there are exactly two components in \mathcal{X} . In that case the IE_{full} count of \mathcal{X} is $3|V| - 6$, which is a trivial upper bound on $\text{rank}(G)$.

When dividing edges restricted to C_i into four parts, the edges in \mathcal{P}_2^i may be Maxwell-dependent in some non-trivial Maxwell-independent set of C_i . Hence we should define \mathcal{M} as $\mathcal{M} = \bigcup_i^m \mathcal{P}_1^i \bigcup_i^m (\mathcal{P}_4^i \cap \mathcal{M}_i)$. In the following we will show that \mathcal{M} is maximal Maxwell-independent.

The major part of the proof that \mathcal{M} is Maxwell-independent is the same as in Theorem 6. However, there is a minor difference when we try to show that \mathcal{M} is maximal Maxwell-independent: when edges in \mathcal{P}_2^i are Maxwell-dependent in some non-trivial Maxwell-independent set of C_i . In that case, we know there exist two components C_i and C_j such that $\mathcal{M}_i \cup \mathcal{M}_j$ is Maxwell-independent with Maxwell count 6. Since \mathcal{X} is a complete collection of proper vertex-maximal components, C_i and C_j are the only two components in \mathcal{X} and $C_i \cup C_j = G$ is also Maxwell-rigid. Thus \mathcal{M} consists of $3|V| - 6$ edges and is a trivial upper bound on $\text{rank}(G)$. \square

Remark: (i) In fact, in Theorems 6 and 7, when G is not Maxwell-rigid or G has more than 3 components in the strong 2-thin cover \mathcal{X} , it turns out that we do not *need* Theorem 1 to show that the IE_{full} count of the cover \mathcal{X} is an upper bound on $\text{rank}(G)$. This is because we can show that \mathcal{M} constructed in Theorem 6 is in fact a *maximum-size* Maxwell-independent set of G . Otherwise we can find a maximal Maxwell-independent set \mathcal{M}' such that $|\mathcal{M}'| > |\mathcal{M}|$. Then there must be some i such that $|\mathcal{M}'|_i > |\mathcal{M}|_i$. We know \mathcal{P}_2^i is Maxwell-independent in every Maxwell-independent set of C_i and since $\mathcal{M}'|_i$ is Maxwell-independent, hence $\mathcal{M}'|_i \cup \mathcal{P}_2^i$ is also Maxwell-independent with size greater than $\mathcal{M}|_i \cup \mathcal{P}_2^i$, which is \mathcal{M}_i . That is a violation to the fact that \mathcal{M}_i is a maximum Maxwell-independent set of C_i . (ii) When G is Maxwell-rigid with exactly two proper vertex-maximal components in a strong 2-thin cover, there are $3|V| - 7$ independent edges in G and we do not need Theorem 1 to show that the IE_{full} count is an upper bound on $\text{rank}(G)$, either. (iii) When \mathcal{X} is a collection of vertex-maximal components, we can use the \mathcal{M} constructed to test Maxwell-rigidity.

5 Open problems

5.1 Extending rank bound to higher dimensions

The definition of maximal Maxwell-independent set extends to all dimensions, leading to the following conjecture.

Conjecture 1. *For any dimension d , the size of any maximal Maxwell-independent set gives an upper bound on the rank of rigidity matroid of a graph G .*

Moreover, the definition of 2-thin component graphs can also be extended to d dimensions.

Definition 10. *Given $G = (V, E)$, let $\mathcal{X} = \{\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_n\}$ be a $(d-1)$ -thin cover of G , i.e., $|\mathcal{M}_i \cap \mathcal{M}_j| \leq d-1$ for all $1 \leq i < j \leq m$. The $(d-1)$ -thin component graph \mathcal{C}_G of G with respect to \mathcal{X} contains a component node for each subgraph induced by \mathcal{M}_i in \mathcal{C}_G and whenever \mathcal{M}_i and \mathcal{M}_j share a complete graph K_{d-1} in G , their corresponding component nodes in \mathcal{C}_G are connected via an edge node. The degree of a component node is defined to be the number of edges incident at it.*

To show Conjecture 1, Observation 1 will have to be shown for $(d-1)$ -thin covers and it is sufficient to show that the $(d-1)$ -thin component graphs of Maxwell-independent sets are generalized partial $\binom{d+1}{2}$ -trees. However, we conjecture one possible generalization of the strongest bound that we are able to show in the proof of Theorem 2(a).

Conjecture 2. *For a Maxwell-independent graph in d dimensions the average degree of the component nodes of any subgraph of a $(d-1)$ -thin component graph with respect to a $(d-1)$ -thin cover \mathcal{X} is strictly smaller than $d+1$.*

In 2D this bound says that for Maxwell-independent sets, the average degree of the component nodes in the component graph is strictly less than 3. In 3D, however, we do not know of an example where all nodes have degree ≥ 3 . In fact, we do not even know of an example with average degree ≥ 3 . We state this as a conjecture for generalized body-hinge frameworks.

Conjecture 3. *In a 3D independent generalized body-hinge framework (where several bodies can meet at a hinge and several hinges can share a vertex), the average number of hinges per body is less than 3.*

5.2 Stronger versions of independence

Even for Maxwell-independent graphs, the rank bounds of our Theorem 1 can be arbitrarily bad. Even a simple example of 2 bananas without the hinge edge has a single maximal Maxwell-independent set of size 18 (which is the bound given by all of our theorems), but its rank is only 17. Another example is the so-called “ n -banana”: it is formed by joining n K_5 ’s on an edge and then remove that shared edge. In the n -banana, the whole graph is Maxwell-independent, so itself is the unique maximal Maxwell-independent set. This maximal Maxwell-independent set exceeds the rank of the 3D rigidity matroid of n -banana by $n-1$.

Theorem 4 and Theorem 5 give better bounds for Maxwell-independent graphs. (In fact, Theorem 5 leads to a recursive method of obtaining a rank bound by recursively decomposing the graph into proper vertex-maximal

components. As one consequence, it gives an alternative, clear proof of correctness for an existing algorithm called the Frontier Vertex algorithm (first version) that is based on this decomposition idea as well as other ideas in this paper such as the component graph [15].)

A natural open problem is to improve the bound in Theorem 1 directly by considering other notions of independence that are stronger than Maxwell-independence. (Algorithms in [16, 15] suggest and use stronger notions than Maxwell-independence, but the algorithms usually use some version of an inclusion-exclusion formula. They do not provide explicit maximal sets of edges satisfying the stronger notions of Maxwell-independence. Neither do they prove that all such sets provide good bounds.)

5.3 Bounds for Maxwell-dependent graphs using 2-thin covers

While Theorems 4 and 5 give strong rank bounds for Maxwell-independent graphs, Theorem 6 and Theorem 7 give much weaker bounds for Maxwell-dependent graphs because the complete collection of (proper) vertex-maximal Maxwell-rigid subgraphs is far from being a 2-thin cover. For example, in Figure 7 we have 3 K_5 's and the neighboring K_5 's share an edge with each other. There are two vertex-maximal, Maxwell-rigid subgraphs, each of which consists of 2 K_5 's with a shared edge.

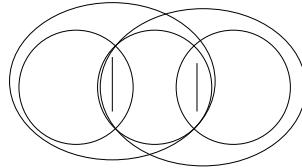


Figure 7: A cover that is not 2-thin. The circles are K_5 's and the two larger ellipses are vertex-maximal, Maxwell-rigid subgraphs that form the cover.

While many other 2-thin covers exist, the vertex-maximality is an important ingredient in the proofs of these theorems. One possibility is to use 2-thin covers that are a subcollection of (proper) vertex-maximal, Maxwell-rigid subgraphs of given graph G . Another is to use collections of not necessarily vertex-maximal, but Maxwell-rigid subgraphs in which no proper subcollection is Maxwell-rigid.

Another notion that can be used involves the following definition of *strong Maxwell-rigidity*:

Definition 11. A graph $G = (V, E)$ is *strong Maxwell-rigid* if for all maximal Maxwell-independent edge sets $E' \subseteq E$, we have $|E'| = 3|V(E')| - 6$.

It is tempting to use the approach in Theorem 6 to show that the IE_{full} count for a cover by vertex-maximal, strong Maxwell-rigid subgraphs is a better upper bound on the rank. We conjecture the 2-thinness of the cover, which is a crucial property explored in proving Theorem 6.

Conjecture 4. Any cover of a graph by a collection of vertex-maximal, strong Maxwell-rigid subgraphs is a 2-thin cover.

However, the idea in the proof of Theorem 6 will not work because the set \mathcal{M} , constructed in the proof of Theorem 6 that is of size equal to the IE_{full} count, can now be of smaller size than any maximal Maxwell-independent set of G as in the example of Figure 8.

Example(Figure 8): there are five rings of K_5 's, where each ring consists of 7 K_5 's. In the graph, every K_5 is a vertex-maximal strong Maxwell-rigid subgraph, and the IE_{full} count for the cover \mathcal{X} is $(3*5-6)*(6*5+1)-5*5-10 = 244$. Here the $(6*5+1)$ is the number of K_5 's and $5*5+10$ is the total number of shared edges. But if we take 9 edges in every K_5 except T such that the missing edges are not shared, then we obtain a set \mathcal{M}' that is Maxwell-dependent. From \mathcal{M}' we drop one edge e of T and add one missing edge f to the K_5 that shares e with T . Then we get a set

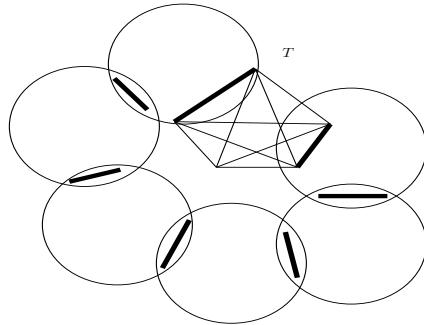


Figure 8: A counterexample to show that IE_{full} count of cover \mathcal{X} by vertex-maximal, *strong* Maxwell-rigid subgraphs turns out to be smaller than the size of any maximal Maxwell-independent set. Start with a K_5 , denoted T . Each of 5 pairs of edges of T is extended into a ring of 7 K'_5 's, where each ring is formed by closing a chain of K_5 's where the neighboring K_5 's share an edge (bold) with each other. In each of the 5 rings, every K_5 shares an edge with each of its two neighboring K_5 's and these two edges are non-adjacent. Note that in the figure, only one of the five rings is shown.

\mathcal{M}'' that is a *minimum-size* maximal Maxwell-independent set of G . The size of \mathcal{M}'' is $(6 * 9 - 5) * 5 = 245$, where $6 * 9 - 5$ is the number of edges in each ring, not counting the edges in T that are unshared in that ring.

Hence in the Figure 8 example, the IE_{full} count is less than the size of any maximal Maxwell-independent set, so the latter cannot be used as a bridging inequality as in Theorem 6. However, the IE_{full} count does give a direct upper bound on the rank in this given example hence a different proof idea might yield the required bound on rank.

5.4 Algorithms for various maximal Maxwell-independent sets

So far the emphasis has been to find good upper bounds on rank and Theorem 1 shows that the *minimum-size* maximal Maxwell-independent set of a graph G is at least $\text{rank}(G)$. A natural open problem is to give an algorithm that constructs a minimum-size, maximal Maxwell-independent set of an arbitrary graph.

Note that Maxwell-rigidity requires the *maximum* Maxwell-independent set to be of size $\geq 3|V| - 6$. Although the maximum Maxwell-independent set is trivially as big as the rank (and is not directly relevant to finding good bounds on rank), covers by Maxwell-rigid components have played a role in some of the Theorems (Theorems 4, 5, 6, 7) that give useful bounds on rank. Recall that Hendrickson [5] gives an algorithm to test 2D Maxwell-rigidity by finding a maximal Maxwell-independent set that is automatically maximum in 2D. While an extension of Hendrickson [5] to 3D given in [13] finds a maximal Maxwell-independent set, it is not guaranteed to be maximum. Thus another question of interest is whether maximum Maxwell-independent sets can be characterized in some natural way.

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